Math 313A **4/1/09**

Test 2 **Due Monday 4/6/09**

Instructions: Write complete and formal proofs for those questions that require proof. **"This is obvious" or "This is a famous theorem" or "We proved this in class" or "This is a theorem in the book", etc. does NOT constitute a proof.** *This is an open notes, open book exam. No additional resources (websites, etc.) or people (classmates, friends, famous or infamous mathematicians, lowly Lynchburg College mathematicians (other than me), etc.) may be used in the completion of this exam.*

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1. Let $w = f(z) = \left(1 - z^2\right)^{\frac{1}{2}}$ and suppose that $w = 1$ when $z = 0$.
   (a) Suppose z starts at the origin and makes one complete counter-clockwise revolution around the circle $|z - 1| = 1$. Find the value of w when z returns to the origin.
   (b) What is the value of w after two rotations around this circle?
   (c) Now suppose z starts at $z = 4$ and makes one complete counter-clockwise revolution around the circle $|z| = 4$. What is the value of w after one revolution?
2. Use the Cauchy-Riemann equations to determine where each of the following functions is differentiable.
   (a) $w = x^2 - y^2 + 3xyi$
   (b) $w = \text{Im}\ (z)$
   (c) $w = |z|$
   (d) $w = x^2 - y^2 + 2xyi$
3. Show that if $f(z)$ is analytic in a domain D and either $\text{Re}\ (f(z))$ or $\text{Im}\ (f(z))$ is constant in D, then $f(z)$ must be constant in D.

4. Let $u(x,y) = 3x(1-y)$.

   (a) Show that $u(x,y)$ is harmonic.

   (b) Find a function $v(x,y)$ such that $f = u + iv$ is analytic.

   (c) Express $f(z)$ in terms of z.

5. Prove that $\sin^2(z) + \cos^2(z) = 1$ for all $z \in \mathbb{C}$ by following the steps below.

   (a) Show that for $z \in \mathbb{C}$ with $z = x + iy$

$$\sin(z) = \sin(x)\left(\frac{e^{-y}+e^{y}}{2}\right) - i\cos(x)\left(\frac{e^{-y}-e^{y}}{2}\right)$$
$$\cos(z) = \cos(x)\left(\frac{e^{-y}+e^{y}}{2}\right) + i\sin(x)\left(\frac{e^{-y}-e^{y}}{2}\right).$$

   (b) Let $f(z) = \sin^2(z) + \cos^2(z)$ and use the Cauchy-Riemann equations to show that $f(z)$ is entire. *(Hint: You really just have to show* $\cos(z)$ *and* $\sin(z)$ *are entire.)*

   (c) Show that $f'(z) = 0$ for all $z \in \mathbb{C}$.

   (d) Show that $f(z)$ is constant.

   (e) Show that $f(0) = 1$.

   (f) Show that $f(z) = 1$ for all $z \in \mathbb{C}$.