Math 601 **Test 1**

Due Monday 6/24/08

Instructions: This is an open notes exam. No additional resources (web sites, etc.) or people (classmates, friends, famous or infamous mathematicians, lowly Lynchburg College mathematicians (other than me), etc.) may be used in the completion of this exam.

I. Decide whether each of the following is true or false. If the statement is true, prove it. If it is false, give a counter-example.

(1) If $a \leq b$ and $c \in \mathbb{R}$, then $ac \leq bc$.

False, this only works if $c \geq 0$. Let $a = 2$ and $b = 3$. Then we have $a \leq b$ but for $c = -1$ we have $-3 \leq -2$, i.e. $bc \leq ac$.

(2) For each $x \in \mathbb{Z}$ there is a unique element $x^{-1} \in \mathbb{Z}$ such that $xx^{-1} = 1$.

False, if $x = 0 \in \mathbb{Z}$, then x^{-1} does not exist. Also if $x = 2 \in \mathbb{Z}$, then $x^{-1} = \frac{1}{2} \notin \mathbb{Z}$.

(3) If $x \in \mathbb{R}$ is not the root of any polynomial with integer coefficients, then $x \notin \mathbb{Q}$.

True.

Proof. We will prove the contrapositive. That is, if $x \in \mathbb{Q}$ then x is the root of a polynomial with integer coefficients. Suppose $x \in \mathbb{Q}$. Then we can write $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ with $q \neq 0$. Then x is a root of the polynomial $qx - p$ which has integer coefficients. □

(4) If $\alpha \notin \mathbb{Q}$, then $\sqrt{\alpha} \notin \mathbb{Q}$.

True.

Proof. To prove the contrapositive, suppose $\sqrt{\alpha} \in \mathbb{Q}$. Then since $\mathbb{Q}$ is closed under multiplication we have $\alpha = (\sqrt{\alpha})^2 \in \mathbb{Q}$. □

(5) If $x \in \mathbb{R}$ is an algebraic number, then it is rational.

False. Let $x = \sqrt{2}$. Then x is a root of $x^2 - 2$ which has integer coefficients, but $x \notin \mathbb{Q}$.

II. Prove each of the following.

(1) Let α be any irrational number and r be any nonzero rational number. Prove that the addition, subtraction, multiplication, and division of r and α yield irrational numbers. That is, Prove that $\alpha + r$, $\alpha - r$, $r - \alpha$, $r\alpha$, $\frac{r}{\alpha}$ and $\frac{\alpha}{r}$ are all irrational numbers.

Proof. Let α be irrational and $r \in \mathbb{Q}$. Suppose $\alpha + r = r_1 \in \mathbb{Q}$. Then $\alpha = r_1 - r$. Since $\mathbb{Q}$ is closed under addition, $r_1 - r \in \mathbb{Q}$ which contradicts $\alpha \notin \mathbb{Q}$.

The other cases are similar since the rationals are closed under all basic operations . □

(2) Let x be an algebraic number and $n \in \mathbb{N}$. Prove that $\sqrt[n]{x}$ is also algebraic.

Proof. Since x is algebraic, it is a root of

$$p_1(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0$$

where $a_i \in \mathbb{Z}$ for $i = 0, 1, \ldots, m$. Then $\sqrt[n]{x}$ is a root of the polynomial $p_2(t) = a_m t^{mn} + a_{m-1} t^{(m-1)n} + \cdots + a_1 t^n + a_0$. This is because

$$\begin{aligned} p_2\left(\sqrt[n]{x}\right) &= a_m \left(\sqrt[n]{x}\right)^{mn} + a_{m-1} \left(\sqrt[n]{x}\right)^{(m-1)n} + \cdots + a_1 \left(\sqrt[n]{x}\right)^{n} + a_0 \\ &= a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 \\ &= p_1(x) = 0. \end{aligned}$$

□

(3) Given that α and β are irrational, but $\alpha+\beta$ is rational, prove that $\alpha-\beta$ and $\alpha+2\beta$ are irrational.

Proof. Let $\alpha, \beta \notin \mathbb{Q}$ with $\alpha+\beta = r_1 \in \mathbb{Q}$. Suppose $\alpha - \beta = r_2 \in \mathbb{Q}$. Then

$$r_1 + r_2 = \alpha + \beta + (\alpha - \beta) = 2\alpha \qquad \text{and} \qquad \frac{r_1 + r_2}{2} = \alpha.$$

Since $\mathbb{Q}$ is closed under the basic operations $\frac{r_1+r_2}{2} \in \mathbb{Q}$ which contradicts $\alpha \notin \mathbb{Q}$.

Now suppose $\alpha + 2\beta = r_3 \in \mathbb{Q}$. Then

$$r_3 - r_1 = \alpha + 2\beta - (\alpha + \beta) = \beta.$$

Since $\mathbb{Q}$ is closed under the basic operations this contradicts $\beta \notin \mathbb{Q}$. ☐

(4) Let $a, n \in \mathbb{N}$. Prove that $\sqrt[n]{a}$ is either irrational or an integer. *(Hint: it's algebraic.)*

Proof. Let $a, n \in \mathbb{N}$. Then $\sqrt[n]{a}$ is a root of the polynomial $p(t) = t^n - a$ which has integer coefficients. By the Rational Zeros Theorem the only possible rational roots of this are the integer divisors of a. If $\sqrt[n]{a}$ is among these divisors, then $\sqrt[n]{a}$ is an integer. If $\sqrt[n]{a}$ is not among these divisors, then it is irrational. ☐

(5) Let $a, b \in \mathbb{R}$. Prove that $|b| < a$ if and only if $-a < b < a$.

Proof. ($\Rightarrow$) Suppose $|b| < a$. We need to show two inequalities: $-a < b$ and $b < a$. The first can be rewritten as $-b < a$. Recall that $-b \leq |b|$ and $b \leq |b|$. Now since $|b| < a$ we have, $-b \leq |b| < a$ which is the first inequality. Similarly, $b \leq |b| < a$ which is exactly the second inequality.

($\Leftarrow$) Now suppose $-a < b < a$. In order to show that $|b| < a$ we must show that $b < a$ and $-b < a$ (since $|b| = b$ or $|b| = -b$). By assumption $b < a$ which is the first required inequality. We also know $-a < b$, which can be rewritten as $-b < a$, which is the second required inequality. ☐