

Keep the following in mind:

Theorem 1. *All bounded monotone sequences converge.*

Theorem 2. *If $(x_n)_{n=1}^\infty$ is a monotone sequence, then $\lim_{n \rightarrow \infty} x_n$ exists in the extended sense (i.e. it might be $\pm\infty$).*

Let $(x_n)_{n=1}^\infty$ be a sequence of real numbers. We define liminf and limsup of $(x_n)_{n=1}^\infty$ as:

$$\begin{aligned}\liminf x_n &= \underline{\lim} x_n = \lim_{k \rightarrow \infty} \inf \{x_n : n > k\} \\ \limsup x_n &= \overline{\lim} x_n = \lim_{k \rightarrow \infty} \sup \{x_n : n > k\}.\end{aligned}$$

If we let

$$z_k = \inf \{x_n : n > k\} \quad \text{and} \quad y_k = \sup \{x_n : n > k\},$$

then $(z_k)_{k=1}^\infty$ is monotone (nondecreasing) and $(y_k)_{k=1}^\infty$ is monotone (nonincreasing). So by the above theorem

$$\underline{\lim} x_n = \lim_{k \rightarrow \infty} z_k \quad \text{and} \quad \overline{\lim} x_n = \lim_{k \rightarrow \infty} y_k$$

always exist in the extended sense (i.e. they might be $\pm\infty$).

Classic examples:

- (1) $x_n = (-1)^n$. Then $\underline{\lim} x_n = -1$ and $\overline{\lim} x_n = 1$.
- (2) $x_n = \begin{cases} 2 + \frac{1}{n} & \text{for } n \text{ even} \\ 1 - \frac{1}{n} & \text{for } n \text{ odd} \end{cases}$. Then $\underline{\lim} x_n = 1$ and $\overline{\lim} x_n = 2$.
- (3) $x_n = \begin{cases} \frac{1}{n} & \text{for } n \text{ even} \\ n & \text{for } n \text{ odd} \end{cases}$. Then $\underline{\lim} x_n = 0$ and $\overline{\lim} x_n = \infty$.

Theorem 3. *Let $(x_n)_{n=1}^\infty$ be a sequence of real numbers. Then $(x_n)_{n=1}^\infty$ converges to x_0 (might be $\pm\infty$) if and only if $\underline{\lim} x_n = x_0 = \overline{\lim} x_n$. (i.e. the liminf and limsup are equal and they are equal to x_0 .)*

Examples:

- (1) Notice that none of the sequences in the above examples converge and that $\underline{\lim} x_n \neq \overline{\lim} x_n$ in each case.
- (2) $x_n = \frac{1}{n}$. Of course, $\lim_{n \rightarrow \infty} x_n = 0$. Notice that $\underline{\lim} x_n = 0$ and $\overline{\lim} x_n = 0$. Why? For each $k \in \mathbb{N}$ we have $z_k = \inf \{\frac{1}{n} : n > k\} = 0$ and $y_k = \sup \{\frac{1}{n} : n > k\} = \frac{1}{k}$. So $\lim_{k \rightarrow \infty} z_k = 0$ and $\lim_{k \rightarrow \infty} y_k = 0$.