Math 407A Homework 3

Fall 2004 Solutions

(1) *Proof.* Let $S = \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$. First we need to show that $\sqrt{2}$ is an upper bound. Suppose $\sqrt{2}$ is not an upper bound. Then there is $x \in S$ with $\sqrt{2} < x$. This implies $2 < x^2$ which contradicts $x \in S$.

Now suppose $\varepsilon > 0$. Then there exists a rational number x with

$$\sqrt{2} - \varepsilon < x < \sqrt{2}.$$

Then $x^2 < 2$, so $x \in S$. This means $\sqrt{2} - \varepsilon$ is not an upper bound for S, so $\sqrt{2}$ is the least upper bound. $\square$

(2) $\sup(-S) = -\inf(S)$

Proof. Let $-S = \{x : -x \in S\}$ and $m = \inf(S)$. We want to show $\sup(-S) = -m$. Suppose $x \in -S$. Then $-x \in S$, so $m \leq -x$, which implies $x \leq -m$, so $-m$ is an upper bound for $-S$.

Now suppose M is any upper bound for $-S$. Then $x \leq M$ for all $x \in -S$, so $-M \leq -x$ for all $x \in -S$. This is the same as saying $-M \leq -x$ for all $-x \in S$, so $-M$ is a lower bound for S. Since $m = \inf(S)$, we must have $-M \leq m$. Then $-m \leq M$, so $-m$ is the least upper bound of $-S$. $\square$

$\inf(-S) = -\sup(S)$

Proof. Let $-S = \{x : -x \in S\}$ and $M = \sup(S)$. We want to show $\inf(-S) = -M$. Suppose $x \in -S$. Then $-x \in S$, so $-x \leq M$, which implies $-M \leq x$, so $-M$ is a lower bound for $-S$.

Now suppose m is any lower bound for $-S$. Then $m \leq x$ for all $x \in -S$, so $-x \leq -m$ for all $x \in -S$. This is the same as saying $-x \leq -m$ for all $-x \in S$, so $-m$ is an upper bound for S. Since $M = \sup(S)$, we must have $M \leq -m$. Then $m \leq -M$, so $-M$ is the greatest lower bound of $-S$. $\square$

(3) Let $A, B \subset \mathbb{R}$ be nonempty. Define

$$A + B = \{z = x + y : x \in A, y \in B\}$$

$$A - B = \{z = x - y : x \in A, y \in B\}.$$

Show that

$$\sup(A + B) = \sup(A) + \sup(B)$$

Proof. Let $A, B \subset \mathbb{R}$ be nonempty and set $M_1 = \sup(A)$ and $M_2 = \sup(B)$. (Notice that if A and B are bounded above, then the Completeness Axiom says M_1 and M_2 are real numbers. Otherwise, one or both of these is ∞ in which case the result follows since the right hand side would be ∞.)

We want to show $\sup(A+B) = M_1 + M_2$. Suppose $z \in A+B$. Then $z = x+y$ where $x \in A$ and $y \in B$. Then

$$z = x + y \leq M_1 + M_2$$

since M_1 and M_2 are the supremums of their respective sets. So $M_1 + M_2$ is an upper bound of $A+B$.

Now let $\varepsilon > 0$. Then there exists $x \in A$ with $M_1 - \frac{\varepsilon}{2} < x < M_1$ since M_1 is the supremum of A. Similarly, there exists $y \in B$ with $M_2 - \frac{\varepsilon}{2} < y < M_2$ since M_2 is the supremum of B. Then $z = x+y$ is in the set $A+B$ and

$$M_1 - \frac{\varepsilon}{2} + M_2 - \frac{\varepsilon}{2} < x + y < M_1 + M_2.$$

So we have found $z \in A+B$ with

$$M_1 + M_2 - \varepsilon < z < M_1 + M_2.$$

So $M_1 + M_2$ is the supremum of $A+B$. □

$$\sup(A-B) = \sup(A) - \inf(B).$$

Proof. We have

$$\begin{aligned} \sup(A-B) &= \sup\left(A + (-B)\right) \\ &= \sup(A) + \sup(-B) && \text{by the previous proof} \\ &= \sup(A) - \inf(B) && \text{by (2)} \end{aligned}$$

□

The analogous results for the infimums are:

$$\inf(A+B) = \inf(A) + \inf(B)$$
$$\inf(A-B) = \inf(A) - \sup(B).$$

(4) Given nonempty subsets A and B of *positive* real numbers, define

$$AB = \{z = xy : x \in A, y \in B\}$$
$$\frac{1}{A} = \{z = \frac{1}{x} : x \in A\}.$$

(a) Show that

$$\sup(AB) = \sup(A)\sup(B).$$

Proof. Let $A, B \subset \mathbb{R}$ be nonempty and set $M_1 = \sup(A)$ and $M_2 = \sup(B)$. (Notice that if A and B are bounded above, then the Completeness Axiom says M_1 and M_2 are real numbers. Otherwise, one or both of these is ∞ in which case the result follows since the right hand side would be ∞.)

We want to show $\sup(AB) = M_1 M_2$. Suppose $z \in AB$. Then $z = xy$ where $x \in A$ and $y \in B$. Then

$$z = xy \leq M_1 M_2$$

since M_1 and M_2 are the supremums of their respective sets. So M_1M_2 is an upper bound of AB.
Now let $\delta > 0$ and let $\varepsilon > 0$ be such that $\varepsilon(M_1 + M_2 - \varepsilon) < \delta$. Then there exists $x \in A$ with $M_1 - \varepsilon < x < M_1$ since M_1 is the supremum of A. Similarly, there exists $y \in B$ with $M_2 - \varepsilon < y < M_2$ since M_2 is the supremum of B. Then $z = xy$ is in the set AB and

$$(M_1 - \varepsilon)(M_2 - \varepsilon) < xy < M_1M_2.$$

This can be rewritten as

$$M_1M_2 - M_1\varepsilon - M_2\varepsilon + \varepsilon^2 < xy < M_1M_2.$$

$$M_1M_2 - \varepsilon(M_1 + M_2 - \varepsilon) < xy < M_1M_2.$$

$$M_1M_2 - \delta < M_1M_2 - \varepsilon(M_1 + M_2 - \varepsilon) < xy < M_1M_2.$$

So we have found $z \in A + B$ with

$$M_1M_2 - \delta < z < M_1 + M_2.$$

So $M_1 + M_2$ is the supremum of $A + B$. □

(b) Show that if $\inf(A) > 0$, then

$$\sup\left(\frac{1}{A}\right) = \frac{1}{\inf(A)}.$$

Proof. Let $m = \inf(A)$. We want to show $\sup\left(\frac{1}{A}\right) = \frac{1}{m}$. Let $z \in \frac{1}{A}$. Then $z = \frac{1}{x}$ where $x \in A$. Then $m \leq x$ and so

$$z = \frac{1}{x} \leq \frac{1}{m}.$$

Thus $\frac{1}{m}$ is an upper bound for $\frac{1}{A}$.
Now suppose M is any upper bound for $\frac{1}{A}$. Then if $z \in \frac{1}{A}$ we have $z = \frac{1}{x}$ where $x \in A$. Then $z = \frac{1}{x} \leq M$ for all $z \in \frac{1}{A}$, so

$$\frac{1}{M} \leq \frac{1}{z} = x$$

for all $x \in A$. Therefore $\frac{1}{M}$ is a lower bound for A. So we must have $\frac{1}{M} \leq m$ which gives us $\frac{1}{m} \leq M$. Thus $\frac{1}{m}$ is the least upper bound for $\frac{1}{A}$. □

(c) Show that if $\inf(A) = 0$, then

$$\sup\left(\frac{1}{A}\right) = +\infty.$$

Proof. We will show that $\frac{1}{A}$ is not bounded above. This will give us that $\sup\left(\frac{1}{A}\right) = +\infty$.
Let $0 < M \in \mathbb{R}$. Then since $\inf(A) = 0$ there exists $x \in A$ with $x < \frac{1}{M}$. This implies $M < \frac{1}{x}$. So we have found $z = \frac{1}{x} \in A$ bigger than M. Since this works for any $0 < M \in \mathbb{R}$ we conclude that $\frac{1}{A}$ is not bounded above. □

(5) Let A and B be nonempty subsets of $\mathbb{R}$. Show that

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$$

Proof. Let $M_1 = \sup(A)$ and $M_2 = \sup(B)$. (We know these real numbers exist by the Completeness Axiom.) Let

$$M_0 = \max\{\sup(A), \sup(B)\} = \max(M_1, M_2).$$

We will show that M_0 is an upper bound for $A \cup B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \leq M_1$, and if $x \in B$, then $x \leq M_2$. So $x \leq \max(M_1, M_2) = M_0$.

To show that M_0 is the least upper bound for $A \cup B$, suppose M is any upper bound for $A \cup B$. Then M is an upper bound for A and B. We must have that $M_1 \leq M$ and $M_2 \leq M$ since these are the corresponding least upper bounds. So the maximum $M_0 = \max(M_1, M_2) \leq M$. Thus M_0 is the least upper bound of $A \cup B$. □

$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$$

Proof. Let $m_1 = \inf(A)$ and $m_2 = \inf(B)$. (We know these real numbers exist by the Completeness Axiom.) Let

$$m_0 = \min\{\inf(A), \inf(B)\} = \min(m_1, m_2).$$

We will show that m_0 is a lower bound for $A \cup B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \geq m_1$, and if $x \in B$, then $x \geq m_2$. So $x \geq \min(m_1, m_2) = m_0$.

To show that m_0 is the greatest lower bound for $A \cup B$, suppose m is any lower bound for $A \cup B$. Then m is an lower bound for A and B. We must have that $m_1 \geq m$ and $m_2 \geq m$ since these are the corresponding greatest lower bounds. So the minimum $m_0 = \min(m_1, m_2) \geq m$. Thus m_0 is the greatest lower bound of $A \cup B$. □