Math 407A **9/28/07**

Test 1 **Due Friday 10/5/07**

Instructions: Write complete and formal proofs for those questions that require proof. **"This is obvious" or "This is a famous theorem" or "We proved this in class" or "This is a theorem in the book", etc. does NOT constitute a proof.** *This is an open notes, open book exam. No additional resources (websites, etc.) or people (classmates, friends, famous or infamous mathematicians, lowly Lynchburg College mathematicians (other than me), etc.) may be used in the completion of this exam.*

---

(1) Decide whether each of the following is true or false. If the statement is true, prove it. If it is false, give a counter-example.
   (a) If $a, b \in \mathbb{Q}$ with $a < b$, then there is $r \in \mathbb{Q}$ with $a < r < b$.
   (b) For each $x \in \mathbb{Z}$ there is a unique element $x^{-1} \in \mathbb{Z}$ such that $xx^{-1} = 1$.
   (c) If $x \in \mathbb{R}$ is not the root of any polynomial with integer coefficients, then $x \notin \mathbb{Q}$.
   (d) If b is an irrational number, then $9 - 3b^2$ is also irrational.
   (e) If x is an algebraic number and $n \in \mathbb{N}$, then $\sqrt[n]{x}$ is also algebraic.
   (f) If S is a nonempty subset of $\mathbb{Q}$ that is bounded above, then $\sup(S) \in \mathbb{Q}$.
   (g) If S is a nonempty subset of $\mathbb{R}$ that is bounded, then $\inf(S) < \sup(S)$.

(2) Use induction to prove that $n^2 + n$ is divisible by 2 for all $n \in \mathbb{N}$.

(3) Use induction to prove that for all $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}.$$

(4) Prove the following.
   (a) If $|x| < \frac{1}{n}$ for every $n \in \mathbb{N}$, then $x = 0$.
   (b) If $|x - 7| < \frac{1}{n}$ for every $n \in \mathbb{N}$, then $x = 7$.
   (c) If $|x + 3| < \varepsilon$ for any $\varepsilon > 0$, then $x = -3$.

(5) A *sequence* of real numbers is an ordered set

$$(x_n)_{n=1}^{\infty} = \{x_n : n \in \mathbb{N}\} = \{x_1, x_2, x_3, \dots\}$$

of real numbers (i.e. it's a set in which the order of the elements matters). Given any sequence $(x_n)_{n=1}^{\infty}$ of real numbers, define two new sequences $(y_k)_{k=1}^{\infty}$ and $(z_k)_{k=1}^{\infty}$ by

$$y_k = \inf\{x_n : n \geq k\} \quad \text{and } z_k = \sup\{x_n : n \geq k\}.$$

Prove that for each $k \in \mathbb{N}$

$$y_{k-1} \leq y_k \leq z_k \leq z_{k-1}.$$