
An *event*, A, is a subset of a sample space S.

Example 1. *If S is the set of outcomes we get from tossing a coin 3 times, then an event, A, could be the set of outcomes with 2 heads. So*

$$A = \{HHT, HTH, THH\}.$$

Let S be a sample space and A and B be events in S. Then A and B are *independent events* if

$$P(A \cap B) = P(A)P(B).$$

In general, the events $A_1, A_2, \ldots, A_n$ are independent if the probability of any $2, 3, 4, \ldots,$ or n of these events is the products of their respective probabilities. That is,

$$\begin{aligned}
P(A_i \cap A_j) &= P(A_i)P(A_j) && \text{for all } 1 \le i \ne j \le n \\
P(A_i \cap A_j \cap A_k) &= P(A_i)P(A_j)P(A_k) && \text{for all } 1 \le i \ne j \ne k \le n \\
P(A_i \cap A_j \cap A_k \cap A_l) &= P(A_i)P(A_j)P(A_k)P(A_l) && \text{for all } 1 \le i \ne j \ne k \ne l \le n \\
&\vdots \\
P(A_1 \cap A_2 \cap \cdots \cap A_n) &= P(A_1)P(A_1)\cdots P(A_n)
\end{aligned}$$

Example 2. *Suppose $A, B,$ and C are events in a sample space S with $P(A) = P(B) = P(C) = \frac{1}{2}$ and $P(A \cap B \cap C) = \frac{1}{4}$. Are the events independent?*

We extend the idea of independence to random variables in a similar way.

Let X and Y be random variables. Then X and Y are *independent random variables* if

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

for all possible values x and y. (This is another way of saying that the events $X = x$ and $Y = y$ are independent events)

Example 3. *Suppose we toss a coin 3 times and write down a 1 if we toss a head and a 0 if we toss a tail. Let* X_1 *be the result of the first toss,* X_2 *be the result of the second toss, and* X_3 *be the result of the third toss. Then*

$$\begin{aligned} P(HTH) &= P(X_1 = 1, X_2 = 0, X_3 = 1) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= P(X_1 = 1)P(X_2 = 0)P(X_3 = 1). \end{aligned}$$

(Notice, these are independent in the sense that the result of any given toss does not affect the result of the next toss, so the subsequent probabilities remain the same.)

We will be interested in a sequence $X_1, X_2, \ldots, X_n$ of independent identically distributed random variables, which means they are independent and all have the same distribution function $f(x)$. Then

$$\begin{aligned} P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) &= P(X_1 = x_1)P(X_2 = x_2)\ldots P(X_n = x_n) \\ &= f(x_1)f(x_2)\ldots f(x_n). \end{aligned}$$

Example 4. *Suppose we toss a standard die 5 times and let the random variable* X_i *be the result of the* i^{th} *toss. That is* $x_i = 1, 2, 3, 4, 5,$ *or* 6 *for each* $1 \leq i \leq 10$. *Then, since the individual tosses are independent, the probability of tossing the sequence* $2, 5, 3, 3, 6$ *is*

$$\begin{aligned} P(X_1 = 2, X_2 = 5, X_3 = 3, X_4 = 3, X_5 = 6) &= P(X_1 = 2)P(X_2 = 5)P(X_3 = 3)P(X_4 = 3)P(X_5 = 6) \\ &= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \\ &= \frac{1}{7776}. \end{aligned}$$

Notice that there are 7776 outcomes in the sample space of 5 consecutive tosses of a die and our sequence is just one of them.

Example 5. *Suppose we look at a population of 12,000 students and, in order to draw conclusions about their heights, we draw a random sample (with replacement) of 100 students and measure their heights.*
This gives us a sequence of independent identically distributed random variables (because we sampled with replacement) $X_1, X_2, \ldots, X_{100}$ *that represent the height of the first, second, ..., hundredth person we selected. We could then calculate the average height for our sample, draw another sample and calculate its average, etc.*

Any value obtained from a sample for the purpose of estimating a population parameter is called a *sample statistic* or just *statistic*. A sample statistic taken from a sample of size n can be thought of as a function of the random variables $X_1, X_2, \ldots, X_n$.

Example 6. *If we draw a sample of 100 students and find their average height, then the sample average is the statistic given by the function*

$$g(X_1, X_2, \ldots, X_n) = \frac{X_1 + X_2 + \cdots + X_{100}}{100}.$$

This function is another random variable, so it has a distribution, mean, variance, etc. which we will study.