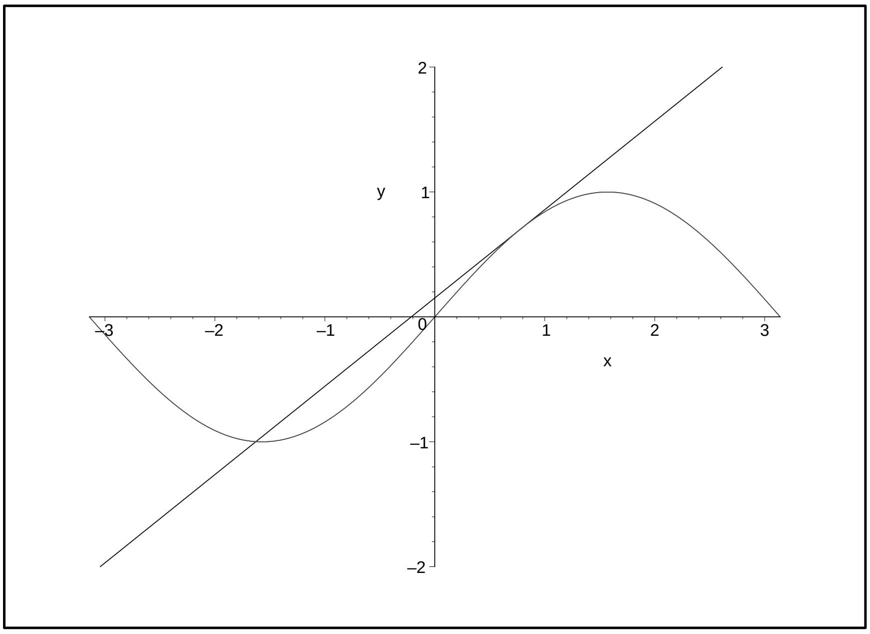
Recall that a function $f(x)$ can be approximated *locally* by its derivative. In other words, for any x near a point a the function value $f(x)$ can be approximated using the function value $f(a)$ and the derivative $f'(a)$.
The equation of the tangent line at the point $(a, f(a))$ is found by

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(a) &= f'(a)(x - a) \\ y &= f(a) + f'(a)(x - a) \end{aligned}$$

So if x is close to a, then the value of this line at x will be close to $f(x)$. (Draw a picture)

Example 1. $f(x) = \sin(x)$ near $a = \frac{\pi}{4}$ can be approximated by the line

$$y = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right).$$



Exercise 1. Find the equation of the tangent line to the function $f(x) = \dfrac{1}{x}$ at the point $a = 2$. Sketch a graph of $f(x)$ and the tangent line.

Note that this tangent line approximation works because the tangent line to $f(x)$ at a is *the only line with slope $f'(a)$ passing through the point* $(a, f(a))$. We can generalize this to second degree approximations by finding the parabola passing through the point $(a, f(a))$ with the same slope (first derivative) as $f(x)$ at a and the same *second* derivative as $f(x)$ at a.

Example 2. Consider $f(x) = e^{-(x-1)}$ at $a = 1$. The parabola we want looks like

$$p(x) = c_1 + c_2(x-1) + c_3(x-1)^2.$$

We look at it in this form because it makes it easy to find the coefficients. We will find c_1, c_2 and c_3 to make the first and second derivatives and the function value at a match up. The derivatives are

$$p'(x) = c_2 + 2c_3(x-1) \qquad p''(x) = 2c_3.$$

Evaluating these (and $p(x)$) at $x = 1$ gives us

$$\begin{aligned} p(1) &= c_1 \\ p'(1) &= c_2 \\ p''(1) &= 2c_3 \end{aligned}$$

Since we want these derivatives to match up with those of f, we need

$$\begin{aligned} f(1) &= c_1 \\ f'(1) &= c_2 \\ f''(1) &= 2c_3 \end{aligned}$$

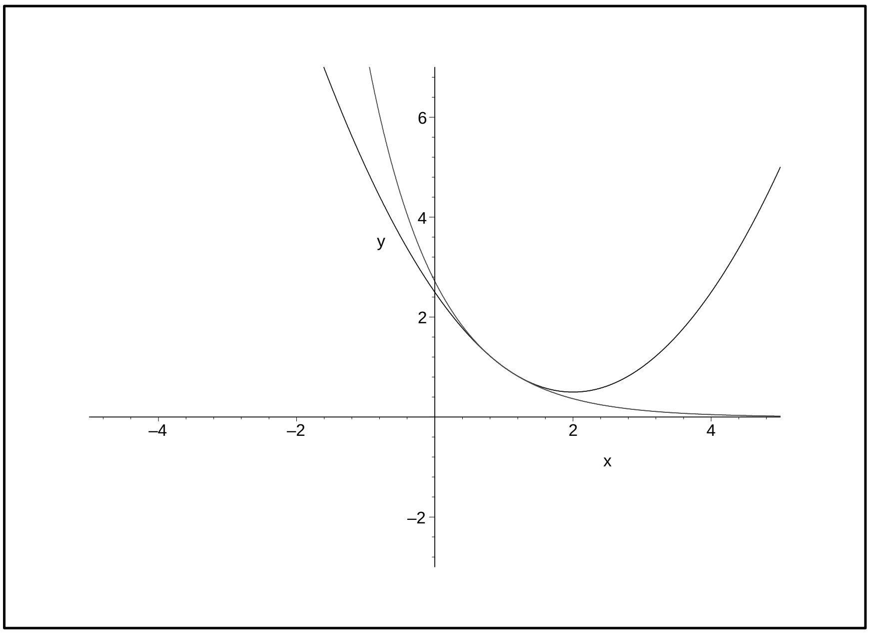
or

$$\begin{aligned} f(1) &= e^{-0} = 1 = c_1 \\ f'(1) &= -e^{-0} = -1 = c_2 \\ f''(1) &= e^{-0} = 1 = 2c_3. \end{aligned}$$

This gives us $c_1 = 1, c_2 = -1$ and $c_3 = \frac{1}{2}$. So our parabola is

$$p(x) = 1 - (x-1) + \frac{1}{2}(x-1)^2.$$

This polynomial is *the second degree Taylor polynomial* of $f(x)$ centered at $a = 1$. Notice that close to 1 this parabola approximates the function rather well.



Using this as a model we can give a general form of the second degree Taylor polynomial for $f(x)$ at a:

$$p(x) = c_1 + c_2(x-a) + c_3(x-a)^2.$$

We need

$$\begin{aligned} f(a) &= c_1 \\ f'(a) &= c_2 \\ f''(a) &= 2c_3 \end{aligned}$$

So we get

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Exercise 2. Find the second degree Taylor polynomial for $f(x) = \frac{1}{x}$ at $a = -2$. Draw a picture.

Higher degree Taylor polynomials are found in the same way. For example, the third degree Taylor polynomial for a function $f(x)$ centered at a is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.$$

In general, the n^{th} degree Taylor polynomial for $f(x)$ at a is:

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The symbol $n!$ (read "n factorial") is used to represent the product of all numbers less than or equal to n. In other words, if n is a positive integer, then n factorial is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 4 \cdot 3 \cdot 2 \cdot 1.$$

Note that $2! = 2$ and $3! = 6$ and we define $0! = 1$.
The n^{th} degree Taylor polynomial is sometimes denoted $T_n(x)$ and we can write it using summation notation as

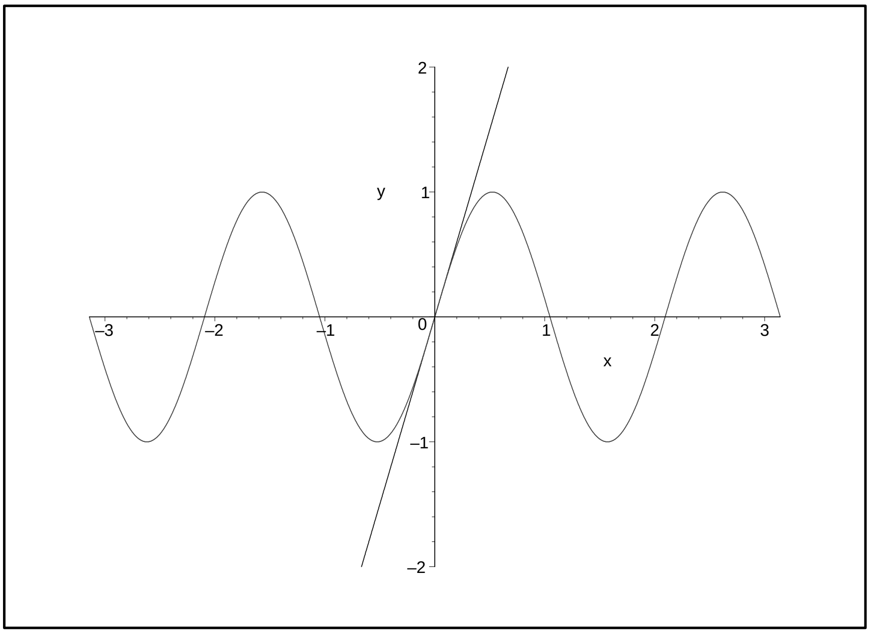
$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Exercise 3. Compute the fifth degree Taylor polynomial (i.e. $T_5(x)$) for $f(x) = \sin(3x)$ centered at zero (i.e. $a = 0$).

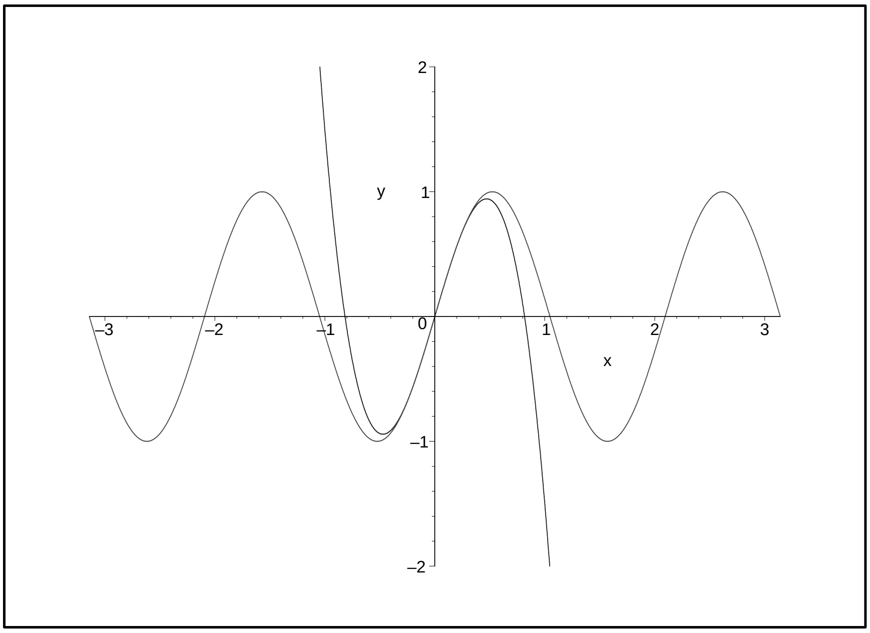
Why are higher degree Taylor polynomials better?

Below are graphs of $f(x) = \sin(3x)$ and several of its Taylor polynomials.

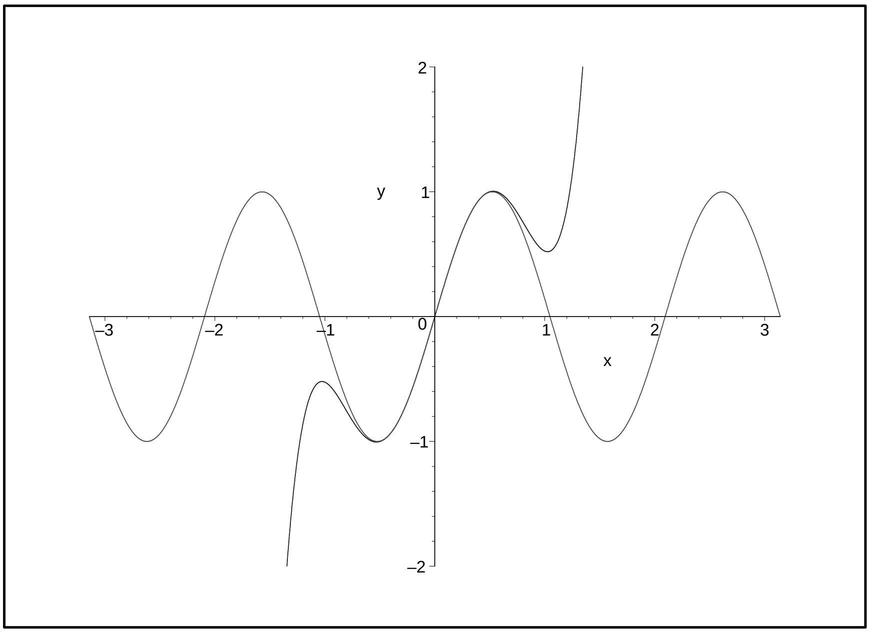
$T_1(x)$



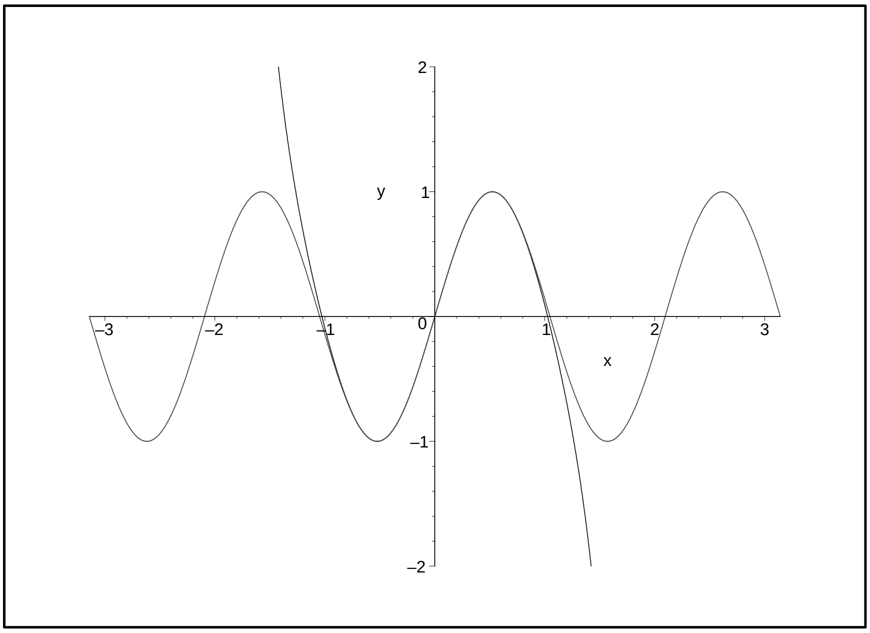
$T_3(x)$



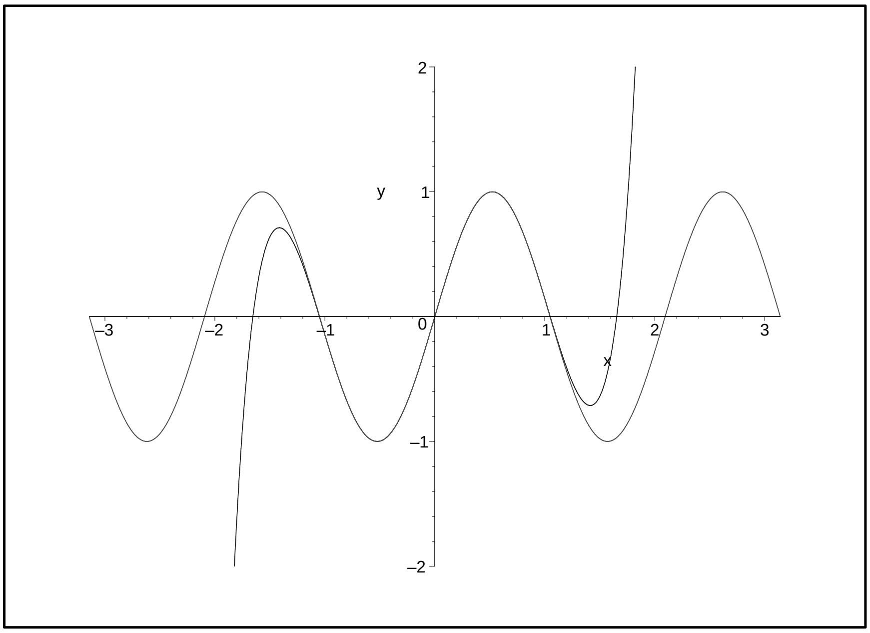
$T_5(x)$



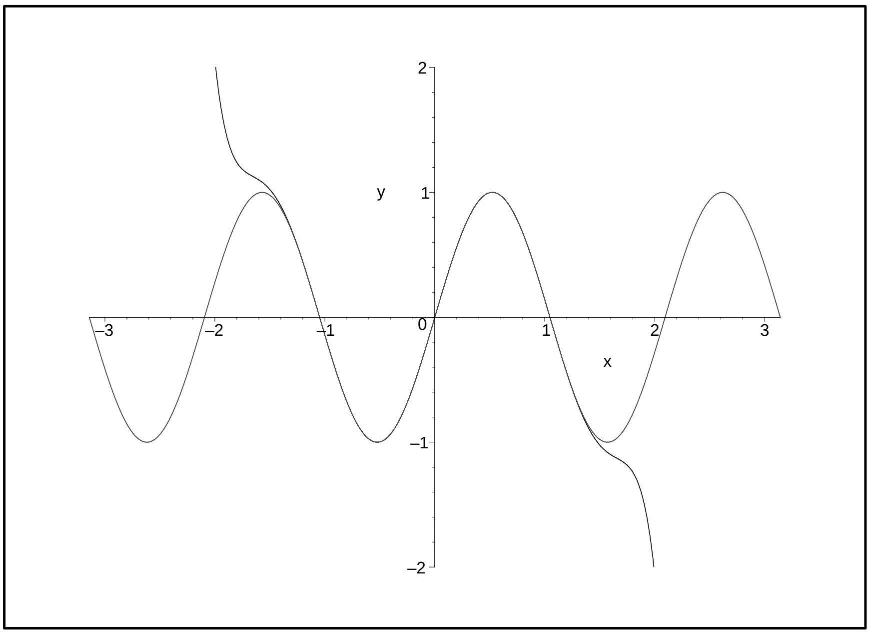
$T_7(x)$



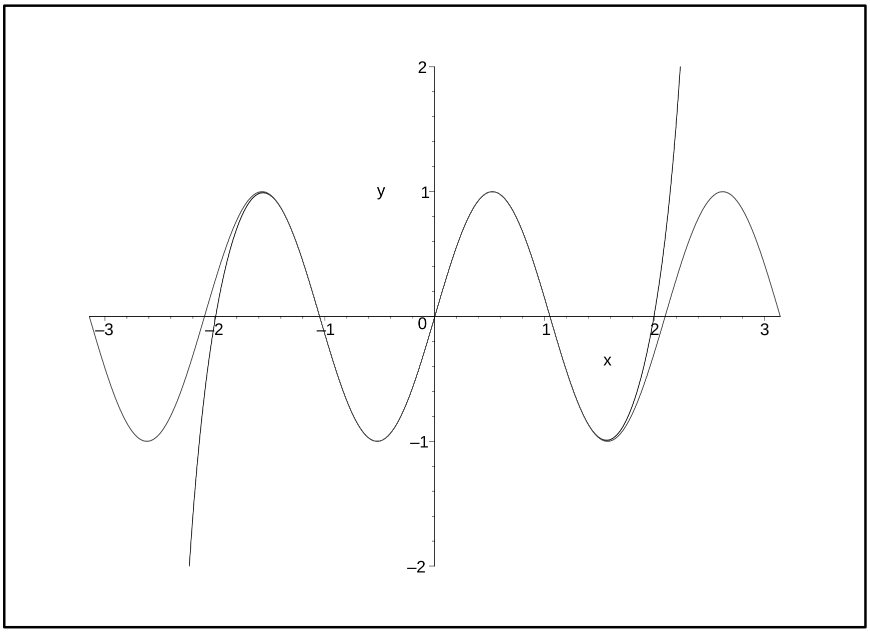
$T_9(x)$



$T_{11}(x)$



$T_{13}(x)$



Exercise 4. What do you notice happening as the degree of the Taylor polynomial increases? Can you use the Taylor polynomials to obtain an approximation of the function value $f(x)$ at any x value? Explain.

Exercise 5. Fill in the equations of the Taylor polynomials above. Do you notice any pattern in the coefficients? Why did we only use odd-degree Taylor polynomials?

Exercise 6. (1) Can you evaluate $\int \cos\left(3x^2\right)\, dx$? Explain.

(2) Evaluate $\int_{-0.4}^{0.4} \cos\left(3x^2\right)\, dx$.
Hint: Calculate the fifth degree Taylor polynomial for $f(x) = \cos(3x^2)$ centered at 0.

(3) Is this the exact value or an approximation? If it is an approximation, explain how you could make it closer.

Exercise 7. (1) Can you evaluate $\int e^{x^2}\, dx$? Explain.

(2) Evaluate $\int_{-0.5}^{0.5} e^{x^2}\, dx$.
Hint: Calculate the fifth degree Taylor polynomial for $f(x) = e^{x^2}$ centered at 0.

(3) Is this the exact value or an approximation? If it is an approximation, explain how you could make it closer.

Exercise 8. What might be the advantage of representing a function by one of its Taylor polynomials? Use the last Taylor polynomial you calculated in the previous problem as an example.