Math 104 **Name:**_____

Final Exam **Due 3:00pm Friday 12/14/07**

Instructions:

-Do not put any work or answers on this sheet. Clearly number all your solutions. In order to receive full credit you must show all your work.

-This is an open notes, open book exam. No additional resources (websites, etc.) or people (classmates, friends, famous or infamous mathematicians, lowly Lynchburg College mathematicians other than myself, etc.) may be used in the completion of this exam.

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1. (a) Calculate the total area bounded by the curve $f(x) = x^2 - 4x + 3$ and the x-axis on the interval $[0, 4]$.

   (b) Calculate $\displaystyle\int_0^4 x^2 - 4x + 3\,dx$

   (c) Are these two things the same? Explain why or why not.

2. Use differentiation to show that $\sin^{-1}(x) = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ for $|x| < 1$.
   (Hint: If $f'(x) = g'(x)$ then $f(x) = g(x) + C$ for some constant C.)

3. Show that $\cos\left(\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx\right) = -1$.

4. The *Two-Point Quadrature Approximation* for $f(x)$ is

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

   (a) Use this formula to approximate $\displaystyle\int_{-1}^{1} \cos(x)\,dx$. Find the error of the approximation.

   (b) Use this formula to approximate $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx$. Find the error of the approximation.

   (c) Show that the Two-Point Approximation is exact for $f(x) = x^2$.

   (d) **Bonus:** Prove that the Two-Point Quadrature Approximation is exact for all polynomials of degree 3 or less.

5. At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 2 grams. The maximum weight of the culture is 10 grams.

   (a) Write a logistic equation that models the weight of the bacterial culture.

   (b) Find the culture's weight after 5 hours.

   (c) When will the culture's weight reach 8 grams?

6. Let L be the tangent line to the graph of the function $y = \ln(x)$ at the point (a, b). Show that the distance between b and the y-intercept of L is always equal to 1.

7. Let L be the tangent line to the graph of the function $y = e^x$ at the point (a, b). Show that the distance between between a and the x-intercept of L is always equal to 1.

8. The formula for the amount A in a savings account earning interest at a rate of r compounded n times per year for t years with an initial deposit of P is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

   Show that the limiting formula as the number of compoundings per year becomes infinite is given by

$$A = Pe^{rt}.$$

9. Suppose that $f(a) = f(b) = g(a) = g(b) = 0$ and the second derivatives of $f(x)$ and $g(x)$ are continuous on the closed interval $[a, b]$. Show that

$$\int_a^b f(x)g''(x)\,dx = \int_a^b f''(x)g(x)\,dx.$$

10. The *Gamma Function* $\Gamma(n)$ is defined for $n > 0$ by

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx.$$

    (a) Find $\Gamma(1)$, $\Gamma(2)$ and $\Gamma(3)$.

    (b) Use integration by parts to show that $\Gamma(n) = (n-1)\Gamma(n-1)$.

    (c) Use the result of part (b) to show that for any positive integer n

$$\Gamma(n) = (n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1.$$

    Note: This says $\Gamma(n) = (n-1)!$.