The Area under a curve

If $f(x)$ is a *continuous nonnegative* function on an interval $[a, b]$, then the area bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area } = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where c_i is any number in the interval $[x_{i-1}, x_i]$ and $\Delta x = \dfrac{b-a}{n}$.

Note:

(1) We require f to be continuous and nonnegative because we're talking about area.

(2) The width of all the rectangles is the same (i.e. Δx). This is called a *regular partition* of $[a, b]$.

Riemann Sums

Let $f(x)$ be defined on the interval $[a, b]$ and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the i^{th} subinterval (i.e. $\Delta x_i = x_i - x_{i-1}$). If c_i is any point in the i^{th} subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

is called a *Riemann Sum* of f for the partition Δ.

Note:

(1) Now there are no restrictions on f. We only require that it be defined on a closed interval.

(2) We can still think of this as the sum of areas of rectangles, but the way we defined the partition Δ does not imply that the width of each rectangle is the same. So we may have varying widths. We will deal only with regular partitions as above. Everything works out the same.

(3) So we can think of a Riemann Sum as

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

where $x_{i-1} \le c_i \le x_i$ and $\Delta x = \dfrac{b-a}{n}$, like before.

Definite Integrals

If $f(x)$ is defined on the closed interval $[a, b]$ and the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

exists, then f is *integrable* on $[a, b]$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

is called the *definite integral* of f from a to b. The numbers a and b are called the *lower* and *upper limits of integration*, respectively.

Note:

(1) We're using a regular partition of $[a, b]$ in this definition (i.e. all rectangles have the same width).

(2) See the book (p.267) for the definition using an arbitrary partition.

(3) The connection will be made between this and indefinite integrals later. Notice that this is a number while an indefinite integral is a collection of functions.

(4) If f is continuous and nonnegative on $[a, b]$, then the area under the curve is

$$\text{Area } = \int_a^b f(x)\,dx.$$

Properties of Definite Integrals

(1) If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.

(2) If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

(3) If f is integrable on the intervals $[a, b]$, $[b, c]$ and $[a, c]$, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

(4) If f and g are integrable on $[a, b]$ and $k \in \mathbb{R}$, then the functions kf and $f \pm g$ are integrable on $[a, b]$ and

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

(5) If f is integrable and nonnegative on $[a, b]$, then $0 \le \int_a^b f(x)\,dx$.

(6) If f and g are integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$