Recall the general power rule for antidifferentiation:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1.$$

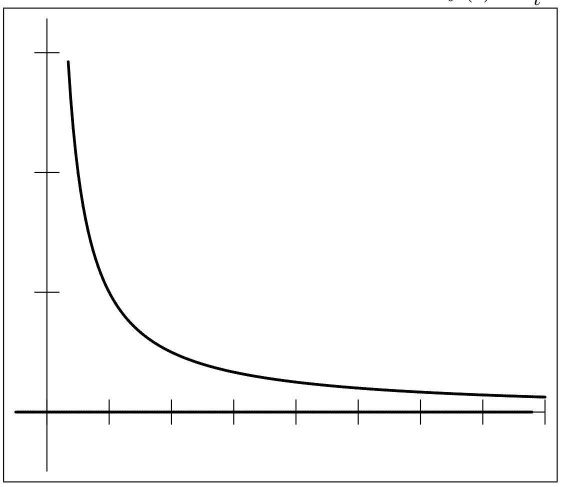
This doesn't work for $n = 1$ because we would divide by 0, so we have to define $\int \frac{1}{x} \, dx$ in a different way.

Definition 0.1. The *natural logarithm function* is defined by

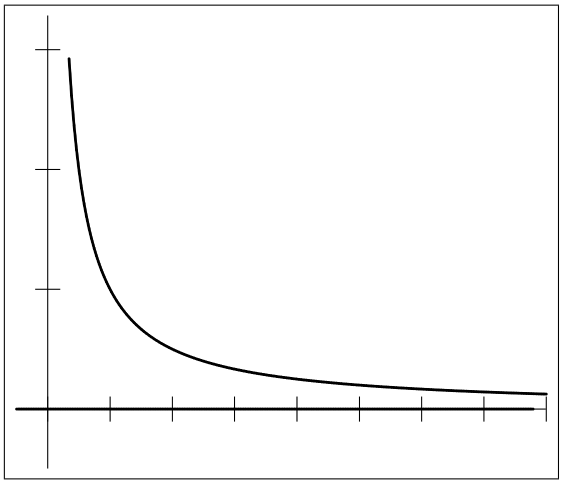
$$\ln(x) = \int_1^x \frac{1}{t} \, dt \quad \text{for } x > 0.$$

Note:

(1) Think of this as the area under the curve $f(t) = \frac{1}{t}$ and draw two pictures.



$0 < x < 1$



$1 < x$

(2) The domain of the natural log function is $(0, \infty)$ and the range is $(-\infty, \infty)$. (Look at the pictures).

(3) This definition is consistent with our previous definition of the natural log function.

Properties:

(1) $\ln(1) = 0$

(2) $\ln(ab) = \ln(a) + \ln(b)$

(3) $\ln(a^n) = n \ln(a)$

(4) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

There is a unique number e such that

$$\ln(e) = \int_1^e \frac{1}{t} \, dt = 1.$$

(Think of the area under the curve.) This is the number e we knew from before ($e \approx 2.71828$)

Derivatives.

By the Second Fundamental Theorem of Calculus we have

$$\frac{dy}{dx}\left[\ln(x)\right] = \frac{dy}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x} \qquad \text{for } x > 0.$$

Now suppose u is a function of x. Then by the chain rule we have

$$\frac{dy}{dx}\left[\ln(u)\right] = \frac{1}{u}\cdot\frac{du}{dx}.$$

Therefore, for any x

$$\begin{aligned}\frac{dy}{dx}\left[\ln|x|\right] &= \begin{cases}\frac{dy}{dx}\left[\ln(x)\right] & \text{if } x > 0\\ \frac{dy}{dx}\left[\ln(-x)\right] & \text{if } x < 0\end{cases}\\ &= \begin{cases}\frac{1}{x} & \text{if } x > 0\\ \frac{1}{-x}(-1) & \text{if } x < 0\end{cases}\\ &= \begin{cases}\frac{1}{x} & \text{if } x > 0\\ \frac{1}{x} & \text{if } x < 0\end{cases}\\ &= \frac{1}{x}\end{aligned}$$

This means $\int \frac{1}{x}\,dx = \ln|x| + C$. This will allow us to integrate many more functions. For example, now we have antiderivatives for all the trig functions:

$$\int \sin(x)dx = -\cos(x) + C \qquad \int \cos(x)dx = \sin(x) + C$$

$$\int \tan(x)dx = -\ln|\cos(x)| + C \qquad \int \cot(x)dx = \ln|\sin(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C \qquad \int \cot(x)dx = -\ln|\csc(x) + \cot(x)| + C$$