
· How to get good at integration? Do LOTS of examples.

· This build your pattern recognition skills

· Always compare the integrals from a new section with those from the old sections to notice differences

· Check answers: $\int F(x)dx = f(x) + C \iff F(x) = f'(x)$ (i.e. take the derivative of your answer and make sure you get the integrand)

Example 1. $\displaystyle\int \frac{1}{x^2+36}\,dx = \frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right) + C$

Check your answer: $D_x\left[\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right) + C\right] = \frac{1}{6}\cdot\frac{1}{1+(\frac{x}{6})^2}\cdot\frac{1}{6} = \frac{1}{36}\cdot\frac{1}{\frac{36+x^2}{36}} = \frac{1}{x^2+36}$

Example 2. $\displaystyle\int \frac{x}{x^2+36}\,dx$

$$u = x^2 + 36$$
$$du = 2x\,dx$$
$$\tfrac{1}{2}du = x\,dx$$

$$\begin{aligned}\int \frac{x}{x^2+36}\,dx = \frac{1}{2}\int \frac{1}{u}\,du &= \frac{1}{2}\ln|u| + C\\ &= \frac{1}{2}\ln|x^2+36| + C\\ &= \frac{1}{2}\ln(x^2+36) + C\\ &= \ln\sqrt{x^2+36} + C\end{aligned}$$

Example 3. *What is wrong with the following?*

See u-substitution above.

$$\begin{aligned}\int_1^2 \frac{x}{x^2+36}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du &= \frac{1}{2}\ln|u|\Big|_1^2\\ &= \frac{1}{2}(\ln(2) - \ln(1))\\ &= \frac{\ln(2)}{2}.\end{aligned}$$

Example 4. *Done correctly:*

See u-substitution above.

$$\int_1^2 \frac{x}{x^2+36}\,dx = \frac{1}{2}\int_{u=37}^{u=40} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{u=37}^{u=40}$$
$$= \frac{1}{2}\left(\ln(40) - \ln(37)\right)$$
$$= \frac{1}{2}\ln\left(\frac{40}{37}\right) = \ln\left(\sqrt{\frac{40}{37}}\right).$$

Example 5. $\displaystyle\int \frac{2x^2+x+75}{x^2+36}\,dx$

If the degree of the top is bigger than the degree of the bottom, then do the division (polynomial long division) first.

$$\frac{2x^2+x+75}{x^2+36} = 2 + \frac{x+3}{x^2+36}$$

$$\int \frac{2x^2+x+75}{x^2+36}\,dx = \int \left[2 + \frac{x}{x^2+36} + \frac{3}{x^2+36}\right]dx$$
$$= 2x + \ln\left(\sqrt{x^2+36}\right) + 3\left[\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)\right] + C$$

Example 6. $\displaystyle\int \frac{dx}{x^2+3x+\frac{5}{2}} = \int \frac{1}{x^2+3x+\frac{5}{2}}\,dx$

This looks like kinda like example 1. Can we make it look more like example 1? Complete the square on the denominator:

$$x^2+3x+\frac{5}{2} = \left(x+\frac{3}{2}\right)^2 + \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \left(x+\frac{3}{2}\right)^2 + \frac{1}{4}$$

$$\int \frac{dx}{x^2+3x+\frac{5}{2}} = \int \frac{1}{\left(x+\frac{3}{2}\right)^2+\frac{1}{4}}\,dx$$
$$= 4\int \frac{1}{(2(x+\frac{3}{2}))^2+1}\,dx$$

$$u = 2(x+\tfrac{3}{2})$$
$$\tfrac{1}{2}du = dx$$

$$= \frac{4}{2}\int \frac{1}{u^2+1}\,du$$
$$= 2\tan^{-1}(u) + C = 2\tan^{-1}\left(2(x+\frac{3}{2})\right) + C$$

Example 7. $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx$

$u = \ln(x)$
$du = \frac{1}{x}\,dx$

$$\int \frac{\cos(\ln(x))}{x}\,dx = \int \cos(u)\,du$$
$$= \sin(u) + C$$
$$= \sin(\ln(x)) + C$$

Example 8. $\displaystyle\int xe^{x^2}\,dx$ *(Recall: $e^{x^2} \neq (e^x)^2 = e^{2x}$)*

Way #1

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}\,du = x\,dx$

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du$$
$$= \frac{1}{2}e^u + C$$
$$= \frac{1}{2}e^{x^2} + C$$

Way #2

$u = e^{x^2}$
$du = e^{x^2}(2x)\,dx$
$\frac{1}{2}\,du = xe^{x^2}\,dx$

$$\int xe^{x^2}\,dx = \frac{1}{2}\int du$$
$$= \frac{1}{2}u + C$$
$$= \frac{1}{2}e^{x^2} + C$$

Example 9. $\displaystyle\int \frac{1}{(x-1)\sqrt{4x^2-8x+3}}\,dx$

Complete the square under the radical:

$$\begin{aligned}\int \frac{1}{(x-1)\sqrt{4x^2-8x+3}}\,dx &= \int \frac{1}{(x-1)\sqrt{4(x-1)^2-1}}\,dx \\ &= \int \frac{1}{(x-1)\sqrt{[2(x-1)]^2-1}}\,dx\end{aligned}$$

$$\boxed{\begin{gathered} u = 2(x-1) \\ du = 2\,dx \\ \tfrac{1}{2}\,du = dx \end{gathered}}$$

$$\begin{aligned} &= \frac{1}{2}\int \frac{1}{\frac{u}{2}\sqrt{u^2-1}}\,du \\ &= \int \frac{1}{u\sqrt{u^2-1}}\,du \\ &= \sec^{-1}(|u|) + C \\ &= \sec^{-1}(|2(x-1)|) + C \end{aligned}$$