
- Suppose $u(x)$ and $v(x)$ are two functions of x. Then by the product rule:

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx} = u'v + uv'.$$

So by the Fundamental Theorem of Calculus:

$$\begin{aligned} uv &= \int u'v + uv'\,dx \\ &= \int v(x)u'(x)\,dx + \int u(x)v'(x)\,dx \\ &= \int v\,du + \int u\,dv. \quad \textit{Note}: du = u'(x)\,dx \textit{ and } dv = v'(x)\,dx \end{aligned}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

- We pick u and dv from the given integrand, then find du and v, then plug in.
- General rules for picking u and dv:
  - → Pick u that is easy to differentiate or that is "better" after differentiating.
  - → Pick dv that is easy to integrate. *Don't forget to include dx in your dv.*

Example 1. $\int xe^x\,dx$

$$\boxed{\begin{array}{ll} u = x & dv = e^x\,dx \\ du = 1\,dx & v = e^x \end{array}}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

If we try the substitution: $\boxed{\begin{array}{ll} u = e^x & dv = x\,dx \\ du = e^x\,dx & v = \frac{x^2}{2} \end{array}}$

$$\int xe^x\,dx = e^x\frac{x^2}{2} - \int \frac{x^2}{2}e^x\,dx$$

which is worse than the original integral (i.e. the power of x is higher).

Example 2. $\int x^2 \cos(x)\, dx$

$u = x^2$	$dv = \cos(x)\, dx$
$du = 2x\, dx$	$v = \sin(x)$

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) - \int 2x \sin(x)\, dx$$

$u = 2x$	$dv = \sin(x)\, dx$
$du = 2\, dx$	$v = -\cos(x)$

$$= x^2 \sin(x) - \left[-2x \cos(x) + \int 2\cos(x)\, dx \right]$$
$$= x^2 \sin(x) + 2x\cos(x) + 2\sin(x) + C.$$

Example 3. $\int e^x \cos(x)\, dx$

$u = \cos(x)$	$dv = e^x\, dx$
$du = -\sin(x)\, dx$	$v = e^x$

$$\int e^x \cos(x)\, dx = \cos(x) e^x + \int e^x \sin(x)\, dx$$

$u = \sin(x)$	$dv = e^x\, dx$
$du = \cos(x)\, dx$	$v = e^x$

$$= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)\, dx$$

This gives us $\int e^x \cos(x)\, dx = \dfrac{e^x\left(\cos(x) + \sin(x)\right)}{2} + C$. *Why?*

Example 4. $\displaystyle\int_0^1 xe^x\,dx$

$u = x \qquad dv = e^x\,dx$ $du = dx \qquad v = e^x$

$$\begin{aligned}
\int_0^1 xe^x\,dx &= xe^x\Big|_0^1 - \int_0^1 e^x\,dx \\
&= xe^x\Big|_0^1 - e^x\Big|_0^1 \\
&= \left[(1)e^1 - (0)e^0\right] - \left[e^1 - e^0\right] \\
&= e - e + 1 = 1
\end{aligned}$$

Example 5. $\displaystyle\int \tan^{-1}(x)\,dx$

$u = \tan^{-1}(x) \qquad dv = dx$ $du = \frac{1}{1+x^2}\,dx \qquad v = x$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx$$

$u = 1 + x^2$ $du = 2x\,dx$ $\frac{1}{2}\,du = dx$

$$\begin{aligned}
&= x\tan^{-1}(x) - \frac{1}{2}\int \frac{1}{u}\,du \\
&= x\tan^{-1}(x) - \frac{1}{2}\ln|1 + x^2| + C.
\end{aligned}$$