Introduction

A differential equation is any equation that contains a derivative. To solve a differential equation means to find a function that satisfies the equation.

Exercise 1. Verify that the function $y = 2\cos(x) + 3\sin(x)$ is a solution to the differential equation $y'' = -y$.

Is this the only solution to this equation? Explain.

Exercise 2. Verify that $y = 7\,(1 + x^2)$ is a solution to the differential equation $(1 + x^2)\,y' - 2xy = 0$.

The *general solution* of a differential equation is a formula representing the family of functions that satisfy the equation. A *specific solution* can be obtained from the general solution if some extra information about the original function is given. This extra information, called an *initial condition*, is usually given in the form of a point that the graph of the original function passes through.

Example 3. Consider the differential equation $y' = x + 2$. Antidifferentiating with respect to x gives us the general solution $y = \frac{x^2}{2} + 2x + C$ where C represents any constant. Note that this represents the family of antiderivatives of $y' = x + 2$ and each of them is a solution of the differential equation.

If, for example, we were also given that the solution we are interested in passes through the point $(2, -6)$ we would obtain a specific solution of $y = \frac{x^2}{2} + 2x - 12$.

Exercise 4. How are the graphs of the general and specific solutions related?

Separable Differential Equations

A differential equation that can be rewritten so that each variable occurs on only one side of the equation is called a *separable differential equation.* These types of equations can be easily solved using antidifferentiation.

Exercise 5. The following equations are easily seen to be separable. Separate the variable and find the general solution.

$$y' = 4 - x$$

$$y' = x\,(1 + y)$$

$$xy + y' = 100x$$

$$\left(1 + x^2\right) y' - 2xy = 0$$

For the second differential equation above, find the specific solution whose graph passes through the point $(0, 2)$.

Exponential Growth and Decay Models

Several of the differential equations in the previous example involve a specific type of relationship between a function and its derivative. If the rate of change of a quantity y is proportional to the value of y, we can write $y' = ky$ for some constant k. This constant is called a *proportionality constant.* We can solve this by separating the variables and antidifferentiating as follows.

$$
\begin{aligned}
y' &= ky \\
\frac{y'}{y} &= k \\
\int \frac{y'}{y}\,dx &= \int k\,dx \\
\ln|y| &= kx + C \\
y &= Ce^{kx} \qquad \text{if } y > 0.
\end{aligned}
$$

(In most applications, the quantity y will be positive.) Are the two constants C above the same?

Example 6. The following are examples of functions $f(x)$ whose rates of change are proportional to the value of $f(x)$.

(1) **Radioactive decay:** The rate of decay of a radioactive material is proportional to the amount of material. (A large amount of material = a lot of decay)
(2) **Population growth:** The birth rate of a population is proportional to the number of people in the population. (More people = more births)
(3) **Newton's Law of Cooling:** The rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium. (Large temperature difference = faster cooling)
(4) **Compound Interest:** The growth os a compound interest bank account is proportional to the amount of money in the account. (More money = more growth from interest)

Exercise 7. Suppose $y' = ky$ for some proportionality constant k.

(1) Interpret what k means in terms of the rate of change of y. What does $k > 0$ signify? What does $k < 0$ signify?
(2) The general solution of the differential equation $y' = ky$ has the form $y = Ce^{kx}$. What is the significance of the constant C in terms of the quantity y?

Example 8. *Growth rate vs. proportionality constant:* Exponential growth models represent populations with a constant growth rate. The growth rate is different from the proportionality constant k that appears in the equations. To see this consider a population that starts with one member and doubles in size every year. The size of the population after x years would be given by $y = 2^x$. The growth rate of this population is 2 because you can get from one year to the next just by multiplying by 2. Notice that $y = 2^x$ is not a solution to the differential equation

$$\frac{dy}{dx} = 2y.$$

However, the differential equation that $y = 2^x$ solves is

$$\frac{dy}{dx} = \ln(2)y$$

which gives us a proportionality constant of $k = \ln(2)$.

If we consider the standard exponential growth model $y = Ce^{kx}$ that arises from

$$\frac{dy}{dx} = ky$$

we get a growth rate of e^k corresponding to a proportionality constant of k.

Example 9. The *half-life* of a radioactive material is the amount of time it takes for half of the material to decay. For example, the half-life of radioactive Radium is 1599 years. After 1599 years (1 half-life), we would have 50% of the initial amount left. After 3198 years (2 half-lives), we would have 25% of the initial amount left. What percentage of the initial material would be left after 2000 years? 5000 years? 10,000 years?

Example 10. Carbon-14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of radioactive carbon, ${}^{14}C$, absorbed by a tree that grew several centuries ago should be the same as the amount of ${}^{14}C$ absorbed by a tree growing today. Suppose a piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. Assume the half-life of ${}^{14}C$ is 5715 years and determine how long ago the tree was burned to make the ancient charcoal.

The rate of decay of ${}^{14}C$ is proportional to the amount remaining, so the amount of ${}^{14}C$ left in the charcoal after t years is represented by

$$y = Ce^{kt}$$

and we just need to find values for C and k. Note that when $t = 0$ we have

$$y = Ce^{k(0)} = C$$

so C is the initial amount of ${}^{14}C$ in the charcoal, call it y_0. Now our formula is

$$y = y_0e^{kt}$$

and we need to find a value for k.

The half-life of ${}^{14}C$ is 5715 years, so after 5715 years we have half of the initial amount left. This allows us to solve for k:

$$\begin{aligned}
y &= y_0e^{kt} \\
\frac{y_0}{2} &= y_0e^{5715k} \\
\frac{1}{2} &= e^{5715k} \\
\ln\left(\frac{1}{2}\right) &= 5715k \\
k &= \frac{\ln\left(\frac{1}{2}\right)}{5715} \approx -0.00012129.
\end{aligned}$$

Now the amount of ${}^{14}C$ left after t years is given by

$$y = y_0e^{-0.00012129t}.$$

In order to determine when the ancient charcoal was burned we want to find the value of t that gives us 15% of the initial value y_0. We solve the following for t:

$$\begin{aligned}
0.15y_0 &= y_0e^{-0.00012129t} \\
0.15 &= e^{-0.00012129t} \\
\ln(0.15) &= -0.00012129t \\
t &= \frac{\ln(0.15)}{-0.00012129} \approx 15641.19.
\end{aligned}$$

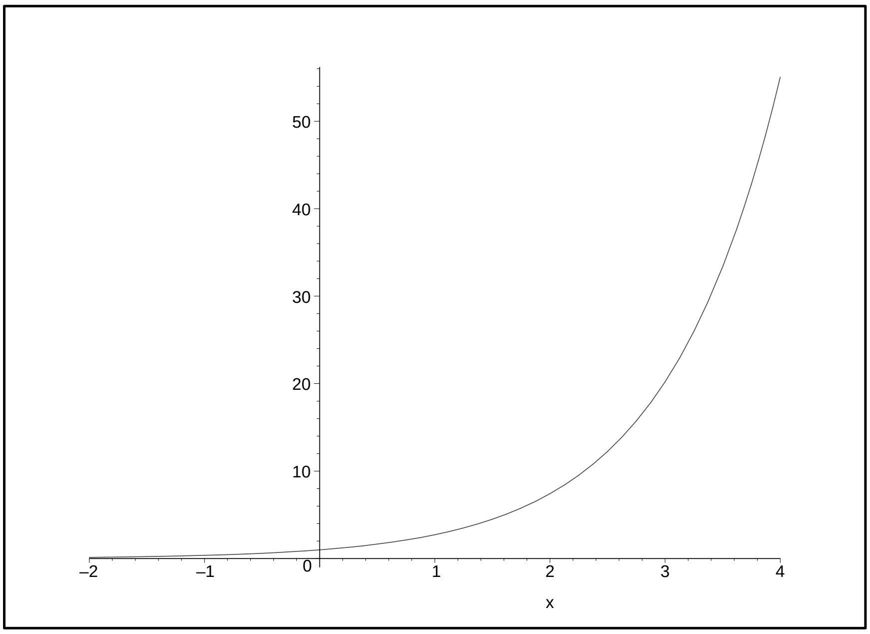
So the ancient charcoal was burned approximately 15,641 years ago. Notice that y_0, the initial amount of ${}^{14}C$ in the tree was irrelevant.

Logistic Growth Equations

We have just seen that if we assume that the rate of change of a population is proportional to the size of the population, then we can represent the size of the population after t years as

$$y = y_0 e^{kt}$$

where y_0 is the initial size of the population and k is the proportionality constant. The graph of this population growth is shown below. Notice that it shows an unbounded growth in the population with respect to time. This is not a very realistic representation of most populations since there are usually some factors that limit the growth of the population.



A more realistic population growth model is given by the *Logistic Growth Equation*, in which the rate of change of the population is proportional to the size of the population and a limiting factor. That is,

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

where the constant L is the upper limit of population growth, called the *carrying capacity of the population.* The value of L is determined by factors such as availability of food, susceptibility to predators, and other factors that might limit population growth. Notice that

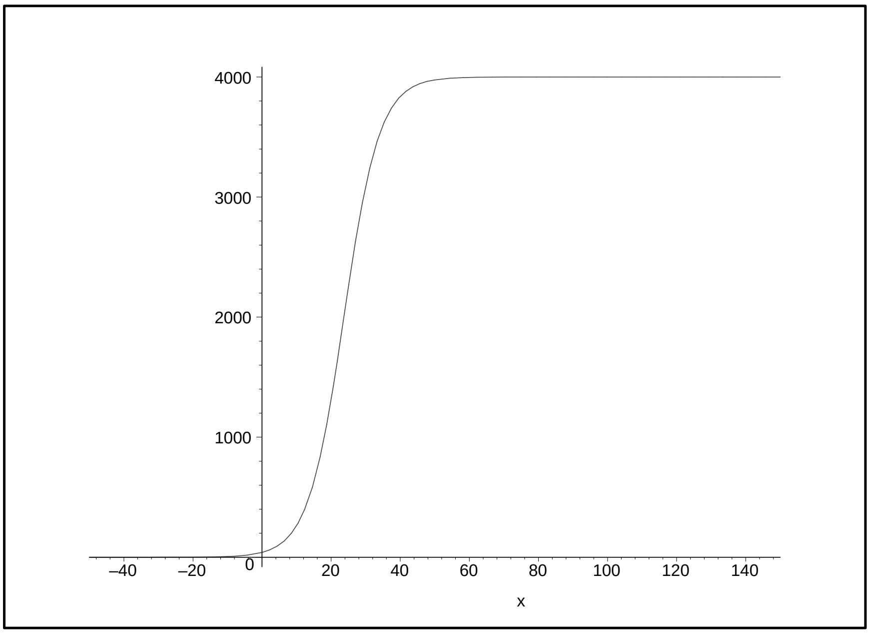
$$\text{if } 0 < y < L, \text{ then } \frac{dy}{dt} > 0$$

$$\text{if } y > L, \text{ then } \frac{dy}{dt} < 0.$$

Exercise 11. Interpret what this means in terms of population growth.

We can find the general solution of the Logistic Growth model in the same way. That is, by separating the variables.

$$\begin{aligned}
\frac{dy}{dt} &= ky\left(1-\frac{y}{L}\right)\\
\frac{1}{y\left(1-\frac{y}{L}\right)}\,dy &= k\,dt\\
\int \frac{1}{y\left(1-\frac{y}{L}\right)}\,dy &= \int k\,dt\\
\int \left(\frac{1}{y}+\frac{1}{L-y}\right)dy &= \int k\,dt\\
\ln|y| - \ln|L-y| &= kt + C\\
\ln\left|\frac{L-y}{y}\right| &= -kt + C\\
\left|\frac{L-y}{y}\right| &= e^{-kt+C} = e^C e^{-kt}\\
\frac{L-y}{y} &= Ce^{-kt} \qquad \text{if } 0 < y < L\\
y &= \frac{L}{1+Ce^{-kt}}.
\end{aligned}$$



Exercise 12. A graph of a typical solution for given values of L and C is shown above.

(1) Describe why this is a more realistic representation of population growth.
(2) Is the population always increasing? Is the growth rate ever negative? Explain.
(3) Is the growth rate of the population always increasing or always decreasing? Explain.
(4) If the growth rate changes from increasing to decreasing, approximate the time at which it changes.

Exercise 13. A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The preserve has an estimated carrying capacity of 200 panthers.

(1) Find the logistic equation that models the population of panthers in the preserve.
(2) Find the population after 5 years.
(3) When will the population reach 100?
(4) At what time is the panther population growing most rapidly? Explain.

The Doomsday Equation

The differential equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

where k and ε are positive constants, is called the *doomsday equation.*

Exercise 14. Solve the doomsday equation

$$\frac{dy}{dt} = y^{1.01}$$

given that $y(0) = 1$. Graph the solution $y(t)$ and find the time T such that $\lim_{t \to T^-} y(t) = \infty$.

This equation is called the doomsday equation because if it is used to model the growth of a population, then it implies that at some finite time T the population will be unbounded (i.e. infinite). In 1960 an electrical engineer at the University of Illinois, Heinz von Foerster, and his colleagues used this model to determine that on November 13, 2026 the population of the world would be unbounded. That is, they predicted that this would be doomsday since the Earth can obviously support an unbounded population. It is interesting to note that this day also happens to fall on a Friday.

This model is based on the assumption that the growth rate of our population is not constant (as the standard exponential model suggests) and is not decreasing (as the Logistic Growth model suggests), but increasing. For example, doubling the population size also doubles the growth rate. The model seemed accurate for several decades following their prediction in the sense that the population continued to rise. In fact, in 1994 two ecologists from the University of Georgia, H. Ronald Pulliam and Nick M. Haddad found that the first time the actual population was less than the value predicted by the doomsday model was in May of 1994. At all times before that the actual population was even greater than the predicted size, which would suggest an even earlier doomsday. This means that the actual growth rate of the population was not always increasing as the model assumed, and the predicted values eventually caught up with the actual population values. Even though the population was always increasing the growth rate was increasing some of the time and then decreasing at other times. This model has since been replaced by more accurate attempts to model the world population. For more information on this and other interesting population models, see How Many People Can the Earth Support? by Joel E. Cohen.