1. (a) Evaluate the integrals: $\displaystyle\int_0^1 \ln(x)\,dx$ $\displaystyle\int_0^1 (\ln(x))^2\,dx$.

   (b) Prove that $\displaystyle\int_0^1 (\ln(x))^n\,dx = (-1)^n n!$ for all positive integers n.
   The expression $n!$ is n *factorial* and represents the product

   $$n! = n(n-1)(n-2)(n-3)\cdots 2\cdot 1.$$

2. Find the value of the positive constant c such that

   $$\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 9.$$

3. In the figure, the segment BD is the height of triangle ΔOAB. Let R be the ratio of the area of ΔDAB to that of the shaded region formed by deleting ΔOAB from the circular sector subtended by angle θ. Find $\lim_{\theta\to 0^+} R$.

4. Show that if $f'(x)$ is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = 0$, then

   $$\int_0^\infty f'(x)\,dx = -f(0).$$

5. A function $f(t)$ is called a *probability density function* if $f(t) \geq 0$ and $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = 1$.

   (a) Interpret what this says about the graph of $f(t)$.

   (b) Show that

   $$f(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

   is a probability density function.

   (c) Show that

   $$f(t) = \frac{1}{\pi(1+t^2)}$$

   is a probability density function.

6. If $f(t)$ is a probability density function, then the probability that x takes on a value less than or equal to b is given by

   $$P(x \leq b) = \int_{-\infty}^{b} f(t)\,dt.$$

As an extension of this we get that if $a \leq b$ then the probability that x takes on a value between a and b is

$$P(a \leq x \leq b) = \int_a^b f(t)\,dt.$$

Let $f(t)$ be the probability density function

$$f(t) = \frac{1}{\pi(1+t^2)}$$

and calculate the probability that x takes on a value less than $\sqrt{3}$.

7. The average value (or mean, or expected value) of x is given by

$$\mu = \int_{-\infty}^{\infty} tf(t)\,dt$$

where $f(t)$ is its probability density function. Find the average value of x with

$$f(t) = \begin{cases} \frac{2}{5}e^{-\frac{2t}{5}} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

8. In 1990 the Ministry of Swallows released a report indicating that the airspeed, x, (measured in meters per second) of an unladen Swallow (European, of course) has the following density function

$$f(t) = \begin{cases} 3e^{-3t} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

   (a) Find the probability that a Swallow will fly at a speed between 1 m/s and 4 m/s.

   (b) On average, what is the airspeed of an unladen (European) Swallow?

9. The *multiplicative derivative* of a function $f(x)$ is given by

$$\widetilde{f}(x) = \lim_{t \to 0} \sqrt[t]{\frac{f(x+t)}{f(x)}}.$$

   Show that if $f(x) = x^n$ then $\widetilde{f}(x) = e^{\frac{n}{x}}$.

10. Explain why the following is true or false: $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\,dx = 0.$